

Groupe de physique des particules, Département de physique,
Université de Montréal, C.P. 6128, succ. centre-ville, Montréal,
Québec, H3C 3J7, Canada

LOW ENERGY NUCLEON-NUCLEON SCATTERING IN THE SKYRME MODEL

T. GISIGER, M.B. PARANJAPÉ

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Abstract

We present the study of the influence of the leading contribution from the kinetic energy of the Skyrme Lagrangian on the scattering of low energy nucleons. This classical kinetic energy for the 2-body system is computed using the product ansatz and is correct for low energy, well separated Skyrmions. We quantize the rotational degrees of freedom of the Skyrmions in order to give them the right spin and isospin values. We are then able to compute analytically the scattering angles for some angular momenta polarizations.

1 Introduction

In this work[1] we present some of the implications of the Skyrme[2] model on the scattering of two nucleons. We will first start with a short introduction to the Skyrme model and its simplest solution, the Skyrmion. Then in the next section we study the problem of the scattering of two Skyrmions using the product ansatz and present the constraints imposed by this parametrization to the scattering processes. The expression of the energy of the two Skyrmion system is then written as an expansion in inverse powers of the separation between the particles. We then briefly describe the approximation methods used to find and solve the equations of motion of the system, namely the method of Lagrange and the method of variation of constants. In

order to study the scattering of nucleons, we have to quantize the remaining degrees of freedom of the system, namely the rotation of each Skyrmon so as to give them the proper spin and isospin. This is done with the Bohr-Sommerfeld quantization technique. We are then able to compute the scattering of two nucleons for certain spin and isospin polarizations.

2 The Skyrme model and the Skyrmon

The Skyrme model was introduced[2] in the 60's and is described by the following Lagrangian:

$$\mathcal{L}_{sk} = -\frac{f_\pi^2}{4} \text{tr}(U^\dagger \partial_\mu U U^\dagger \partial^\mu U) + \frac{1}{32e^2} \text{tr}([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2) \quad (1)$$

where $U(x^\mu)$ is an element of $SU(2)$ which represents a pseudoscalar massless particle, namely the pion. f_π and e are phenomenological parameters related to meson decay and low energy scattering, and are theoretically computable using QCD.

The simplest stable solution of the Skyrme model is the Skyrmon:

$$U_s(\vec{x}) = e^{iF(r)\vec{\tau} \cdot \hat{r}} \quad (2)$$

where $\vec{\tau} = (\tau^1, \tau^2, \tau^3)$ are the three Pauli matrices and the function $F(r)$ is a monotonous function of r which decreases from π at the origin, to 0 at infinity. $F(r)$ has to be computed numerically but falls off at large distance like κ/r^2 . Since the Skyrmon field attains the $SU(2)$ identity when $r \rightarrow \infty$, we can consider the three dimensional space R^3 to be topologically compactified into a 3-sphere S^3 . $U_s(\vec{r})$ then defines an application from an S^3 to the manifold S^3 of the $SU(2)$ group. This type of application is classified into disjoint homotopy classes labeled by the winding number of the first 3-sphere onto the second, N , defined by

$$N = \frac{1}{24\pi^2} \int d^3\vec{x} \epsilon^{ijk} \text{tr}(U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U). \quad (3)$$

An application belonging to one class cannot be deformed continuously into another: it is topologically stable. The Skyrmon has winding number 1, rendering it stable against deformation into the vacuum

defined by $N = 0$. It was Skyrme who first interpreted N as the baryon number.

The Skyrmion is a static solution. In order to set it into motion, we shift its position by the time dependent vector $\vec{R}(t)$ and conjugate it by the time dependent $SU(2)$ matrix $A(t)$:

$$U_1(\vec{r}, t) = A(t)U_s(\vec{x} - \vec{R}(t))A^\dagger(t). \quad (4)$$

We then replace $U_1(\vec{r}, t)$ into the Skyrme Hamiltonian density and integrate over all space. This gives the following expression for the energy:

$$E_1 = M + \frac{M}{2}\dot{\vec{R}}^2 + 2\Lambda\mathcal{L}^a(A)\mathcal{L}^a(A) \quad (5)$$

where we defined the generators of the right and left action of the $SU(2)$ group:

$$\mathcal{R}^a(A) = -\frac{i}{2} \text{tr}[\tau^a A^\dagger \partial_0 A] \quad (6)$$

$$\mathcal{L}^a(A) = -\frac{i}{2} \text{tr}[\tau^a \partial_0 A A^\dagger]. \quad (7)$$

E_1 has the familiar form of the energy of a translating and spinning spherically symmetric rigid body if M , of the order of 1 GeV, is interpreted as the mass, and Λ as the moment of inertia. The last term is the rotational energy, with $\mathcal{L}^a(A)\mathcal{L}^a(A)$ equal to the square of the angular velocity. $\mathcal{L}^a(A)$ is usually interpreted as the tensorial part of the isospin of the Skyrmion, and $\mathcal{R}^a(A)$ as its spin.

It was shown by Atkins, Nappi and Witten[3], that by quantizing the rotational degrees of freedom $A(t)$ the characteristics of the nucleon could be reproduced with an error varying between 10% and 30%. This guides us towards investigating the predictions of the Skyrme model in the baryon number 2 sector.

3 The Skyrmion-Skyrmion interaction

We will now attempt to compute the classical Lagrangian describing a pair of Skyrmons in order to study their scattering. To achieve this we need a parametrization of the pair of Skyrmons. The simplest is the so-called “product ansatz” where the Skyrmon fields are multiplied together:

$$\begin{aligned} U_2(\vec{r}, t) &= U_1(\text{Skyrmion } 1) U_1(\text{Skyrmion } 2) \\ &= A(t)U_s(\vec{x} - \vec{R}_1(t))A^\dagger(t) B(t)U_s(\vec{x} - \vec{R}_2(t))B^\dagger(t). \end{aligned} \quad (8)$$

A and B are $SU(2)$ matrices representing the iso-orientation of the Skyrmions and \vec{R}_1 and \vec{R}_2 their position. This parametrization is not completely general and we have to restrict the initial conditions in order to obtain physically correct results.

First, it is a well known result from numerical studies of the Skyrme model[4] that Skyrmions deform when they come close to each other. In fact, the bound state of two Skyrmions, putatively the deuteron, has the form of a torus. These deformations cannot be described by Eq. 4, let alone the product ansatz, so we have to consider only configurations where the distance $d = |\vec{R}_1(t) - \vec{R}_2(t)|$ is large.

Second, it was shown by N.S. Manton[5] that only low energy soliton systems can be described with a finite number of degrees of freedom. We refer the interested reader to his article and will only say here that it is physically evident that the number of degrees of freedom of a system rises with its energy. Then if we keep the energy of the system sufficiently low, the system might excite only a finite number of modes. If we wish to describe our system with the 12 degrees of freedom of the product ansatz (6 to define the positions of the particles, and another 6 for their iso-orientations), we have to consider only low energy scattering.

We now give the expression of the energy for the pair of Skyrmions, in the center of mass frame, as an expansion in inverse powers of the relative distance d which is assumed to be large at all times:

$$T = \frac{1}{4}M\dot{d}^2 + 2\Lambda(\mathcal{L}^a(A)\mathcal{L}^a(A) + \mathcal{L}^a(B)\mathcal{L}^a(B)) + \frac{\Delta}{d}\epsilon^{iac}\epsilon^{jbd}\mathcal{R}^c(A)\mathcal{R}^d(B)(\delta^{ij} - \hat{d}^i\hat{d}^j)D_{ab}(A^\dagger B) + O(1/d^2). \quad (9)$$

$\Delta = 2\pi\kappa^2 f_\pi^2$ is a result of the integration over space, $\hat{d} = \vec{d}/d$ and $D_{ab}(G) = 1/2 \text{tr}[\tau^a G \tau^b G^\dagger]$ is the 3×3 representation of the $SU(2)$ matrix G . This term, induced by the kinetic energy of the Skyrme Lagrangian, was found by us[6] and independently by B.J. Schroers[7].

We note that this term has a structure similar to the spin-spin and tensorial terms found in the traditional nuclear potentials. Also, the leading contribution from the potential part of the Skyrme Lagrangian behaves as $1/d^3$, thus is neglected to leading order of our expansion of the energy. The kinetic energy defines a metric on the tangent space of the moduli space, the motion implied by the Lagrangian simply follows the geodesics of this metric. This is what we call the geodetic approximation.

4 Equations of motion of the system and quantization

We now have to find the equations of motion corresponding to the Lagrangian of Eq. 9. Because of the very complicated form of the interaction term, we have to resort to various approximation methods. We first choose the observables of the system, meaning, what we need to know from the scattering in order to possibly compare the results with experimental data. We choose the relative velocity \vec{d} , the tensorial part of the spins $\mathcal{R}^a(A)$ and $\mathcal{R}^a(B)$, and isospins $\mathcal{L}^a(A)$ and $\mathcal{L}^a(B)$.

The approximation method of Lagrange[8] is perfectly suited to compute the time derivatives of the observables without having first to derive the equations of motion of A , B and \vec{d} . The method uses the Poisson brackets formalism and is based on the following principle. Let us consider the degree of freedom q^i . Conjugate to q^i is the momentum p^i which can be written as an expansion in powers of $1/d$:

$$p^i = p_0^i + \delta p^i(1/d). \quad (10)$$

Then the Poisson brackets, which are functions of the q^i and the p^i , also form an expansion in powers of $1/d$. If we denote by C^k the observables of the system, the time derivative of C^k is

$$\frac{d}{dt}C^k = \{C^k, H\} = \{C^k, H_0 + H_I\} \quad (11)$$

where we have divided the Hamiltonian into a free H_0 (of order $1/d^0$) and interaction H_I part (of order $1/d$ and higher). By dividing also the Poisson bracket into a free and interaction part, we get

$$\frac{d}{dt}C^k = \{C^k, H_0\}_I + \{C^k, H_I\}_0 \quad (12)$$

plus higher order terms. By computing the various Poisson brackets

involved, we get the following set of 5 coupled equations:

$$\begin{aligned}
\frac{d}{dt}\hat{d}^k &= -\frac{2\Delta}{Md^2} \left[\delta^{ij}\hat{d}^k + \delta^{jk}\hat{d}^i + \delta^{ik}\hat{d}^j - 3\hat{d}^i\hat{d}^j\hat{d}^k \right] \epsilon^{iac}\epsilon^{jbd}\mathcal{R}^c(A)\mathcal{R}^d(B)D_{ab}(A^\dagger B) \\
\frac{d}{dt}\mathcal{L}^k(A) &= \frac{\Delta}{2Md}\epsilon^{iac}\epsilon^{jbd}\mathcal{R}^c(A)\mathcal{R}^d(B)(\delta^{ij} - \hat{d}^i\hat{d}^j)\epsilon^{kef}D_{fa}(A)D_{eb}(B) + \dots \\
\frac{d}{dt}\mathcal{L}^k(B) &= \frac{\Delta}{2Md}\epsilon^{iac}\epsilon^{jbd}\mathcal{R}^c(A)\mathcal{R}^d(B)(\delta^{ij} - \hat{d}^i\hat{d}^j)\epsilon^{kef}D_{ae}(A^\dagger)D_{fb}(B) + \dots \\
\frac{d}{dt}\mathcal{R}^k(A) &= -\frac{\Delta}{2Md}\epsilon^{iac}\epsilon^{jbd}\mathcal{R}^d(B)(\delta^{ij} - \hat{d}^i\hat{d}^j) \\
&\quad \times \left[\epsilon^{kef}\mathcal{R}^f(A)D_{ab}(A^\dagger B) + \epsilon^{kaf}D_{fb}(A^\dagger B)\mathcal{R}^c(A) \right] + \dots \\
\frac{d}{dt}\mathcal{R}^k(B) &= -\frac{\Delta}{2Md}\epsilon^{iac}\epsilon^{jbd}\mathcal{R}^c(A)(\delta^{ij} - \hat{d}^i\hat{d}^j) \\
&\quad \times \left[\epsilon^{kdf}\mathcal{R}^f(B)D_{ab}(A^\dagger B) + \epsilon^{kbf}D_{af}(A^\dagger B)\mathcal{R}^d(B) \right] + \dots
\end{aligned} \tag{13}$$

where the dots represent other complicated terms. This is still too complicated to solve algebraically and we have to resort to further approximations. We choose the approximation method of the variation of constants[8]. Let us consider as an example the following equation:

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), t) \tag{14}$$

and treat it not as an ordinary differential equation but more like an iteration equation:

$$\frac{d}{dt}\vec{x}(t) \simeq f(\vec{x}_0(t), t) \tag{15}$$

where $\vec{x}_0(t)$ is a trial function. Then if the trial function $\vec{x}_0(t)$ is close to the true solution $\vec{x}(t)$ or if $f(\vec{x}_0(t), t)$ is small, then a good estimation of the solution of the equation is:

$$\vec{x}(t) = \int_{-\infty}^t f(\vec{x}_0(t'), t')dt'. \tag{16}$$

In our case, $\vec{x}_0(t)$ is the free value of the C^k , so without the interaction term. The approximation method should give accurate results if the C^k are slowly changing quantities. This is compatible with our working hypothesis. This method decouples the 5 equations of motion and trivializes the integration over time of the observables: all we have to do is replace in each right hand side the free values for \vec{d} , \vec{d} , $\mathcal{R}^a(A)$, $\mathcal{R}^a(B)$, $\mathcal{L}^a(A)$, $\mathcal{L}^a(B)$, A and B , and integrate over time each equation separately.

We will not consider here the time evolution of the spins and isospins. We will only say that spin-flips and isospin-flips seem to occur, which are consistent with spin 1 and isospin 1 particle exchange. We will only consider from now on the equation for the relative momentum $\vec{p} = M/2\vec{\dot{d}}$:

$$\frac{d}{dt}p^i = -\frac{\Delta}{d^2} \left[\delta^{ij} \hat{d}^k + \delta^{jk} \hat{d}^i + \delta^{ik} \hat{d}^j - 3\hat{d}^i \hat{d}^j \hat{d}^k \right] \epsilon^{iac} \epsilon^{jbd} \mathcal{R}^c(A) \mathcal{R}^d(B) D_{ab}(A^\dagger B) \quad (17)$$

in order to extract the scattering angles.

To summarize, we now only have the equation for \vec{p} to solve. This is done by replacing in the right hand side of the equation the free values of \vec{d} and the matrices A and B which define also the spin and isospin of the Skyrmions, and to integrate over time.

So far we have only talked about classical scattering. If we want to study the physics of nucleons, we have to quantize the remaining degrees of freedom, *i.e.* the rotation of each Skyrmion. This means finding the matrices A and B which give the Skyrmions spin and isospin 1/2. The Bohr-Sommerfeld method is perfectly suited to this task. In the well known problem of the hydrogen atom, the sum of the action variables

$$J_i = \oint p_i dq_i \quad (18)$$

is set equal to a multiple of the Plank constant h . This fixes the allowed values for the angular and linear velocities, thereby quantizing the energy and angular momenta of the states. In our case, the same program is applied to the time derivatives of the angles with which the matrices A and B are expressed. Here are the main steps of this procedure.

Let us consider the Lagrangian for a single Skyrmion and express the matrix A of its iso-orientation as a function of 3 Euler angles α , β and γ :

$$A = a_0 + i\vec{a} \cdot \vec{\tau} = e^{-i\alpha\tau_3/2} e^{-i\beta\tau_2/2} e^{-i\gamma\tau_3/2} \quad (19)$$

giving

$$L = -M + \frac{1}{2}\Lambda [\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 + 2\dot{\alpha}\dot{\gamma} \cos \beta]. \quad (20)$$

The equations of motion are satisfied if α and γ are linear functions of time t and if β is equal to 0 or π . The action variables for the angles

α and γ are then readily computed and give:

$$\begin{aligned} J_\alpha &\equiv \oint p_\alpha d\alpha = 2\pi\Lambda[\dot{\alpha} + \dot{\gamma} \cos \beta] = -4\pi\Lambda\mathcal{L}^3(A) \equiv -2\pi I_3 \\ J_\gamma &\equiv \oint p_\gamma d\gamma = 2\pi\Lambda[\dot{\gamma} + \dot{\alpha} \cos \beta] = -4\pi\Lambda\mathcal{R}^3(A) \equiv 2\pi S_3 \end{aligned} \quad (21)$$

using Eq. 6 and Eq. 7, and following the convention used by Adkins[3] *et al.* for the component of the spin and isospin along the z axis

$$\begin{aligned} S_3 &= 2\Lambda\mathcal{R}^3(A) \\ I_3 &= -2\Lambda\mathcal{L}^3(A) \end{aligned} \quad (22)$$

respectively. The Bohr-Sommerfeld quantization condition states that the sum $J_\alpha + J_\gamma$ is equal to an integer multiple of h . This fixes the value of the angular speeds $\dot{\alpha}$ and $\dot{\gamma}$ so as to give, in our particular case, spin and isospin $\pm 1/2$. The expression for the matrix A giving the proper quantum numbers is then obtained by extracting the values of $\alpha(t)$ and $\gamma(t)$ from Eq. 21, after choosing the value of β which is either 0 or π . If $\beta = 0$, solving for the angles α and β gives the matrix

$$A = e^{i(\omega t + \phi_0)\tau^3/2}. \quad (23)$$

ω is the quantized quantity here, and has an absolute value of roughly 100 MeV for the nucleon. This matrix represents the state $|p \downarrow\rangle$ if $\omega > 0$ and $|n \uparrow\rangle$ if $\omega < 0$. If $\beta = \pi$, then we obtain

$$A = -ie^{-i(\omega t + \psi_0)\tau^3/2} \tau^2 e^{i(\omega t + \psi_0)\tau^3/2} \quad (24)$$

which represents the state $|p \uparrow\rangle$ if $\omega > 0$ and $|n \downarrow\rangle$ if $\omega < 0$.

Before going further we note that the time dependence of the matrices A when $\beta = 0$ or $\beta = \pi$ are very different. This will have important consequences on the nucleon-nucleon scattering.

5 Nucleon-Nucleon Scattering

We now have all the necessary tools to solve our problem. To solve Eq. 17 for a particular scattering process, we only replace the corresponding matrices A and B in the right hand side of the equation, as well as the free value of \vec{d} :

$$\vec{d} = (vt, \gamma, 0) \quad (25)$$

which describes a relative particle traveling along the x axis but with an impact parameter γ along the y axis. To respect the restrictions underlined in section 3, v is chosen small to guaranty low energy, and γ large, so as to keep the particles far from each other. In what follows, we choose z as the axis of polarization of the angular momenta. This gives 2-dimensional trajectories. Other polarization axes give complicated 3-dimensional motion best studied numerically.

Since in our approximation scheme the spins and isospins are constant, the only time dependence on the right hand side of Eq. 17 comes from d , \hat{d} and $D_{ab}(A^\dagger B)$. It is then predictable that scattering processes separate into two cases, depending on whether or not $A^\dagger B$ is time independent.

$D_{ab}(A^\dagger B)$ **time independent:**

$$\begin{array}{l} p \uparrow p \uparrow \\ n \downarrow n \downarrow \\ p \downarrow p \downarrow \\ n \uparrow n \uparrow \end{array} \quad \frac{d}{dt} p^k = -\frac{\Delta\omega^2}{d^2} \cos(2\delta) \hat{d}^k \quad (26)$$

$$\begin{array}{l} p \uparrow p \downarrow \\ n \uparrow n \downarrow \end{array} \quad \begin{array}{l} \frac{d}{dt} p^k = -\frac{\Delta\omega^2}{d^2} [\hat{d}^k + 4r^k \hat{r} \cdot \hat{d} - 6\hat{d}^k (\hat{r} \cdot \hat{d})^2] \\ \hat{r}^k = (-\sin(\delta), \cos(\delta), 0) \end{array} \quad (27)$$

$$\begin{array}{l} n \uparrow n \downarrow \end{array} \quad \begin{array}{l} \frac{d}{dt} p^k = -\frac{\Delta\omega^2}{d^2} [\hat{d}^k + 4r^k \hat{r} \cdot \hat{d} - 6\hat{d}^k (\hat{r} \cdot \hat{d})^2] \\ \hat{r}^k = (\sin(\delta), -\cos(\delta), 0) \end{array} \quad (28)$$

where $\delta = \phi_0^A - \phi_0^B$. The scattering angle defined as the angle between the initial ($t = -\infty$) and final ($t = +\infty$) momenta is the same for all these different processes and readily computed:

$$\begin{aligned} \cos \theta &= \frac{\vec{p}(+\infty) \cdot \vec{p}(-\infty)}{|\vec{p}(+\infty)| |\vec{p}(-\infty)|} \\ &= \frac{M\gamma v^2}{4\Delta} \frac{1}{\left(\frac{M^2 \gamma^2 v^4}{16\Delta^2} + \omega^4 \cos^2 2\delta\right)^{1/2}}. \end{aligned} \quad (29)$$

$D_{ab}(A^\dagger B)$ **time dependent:**

$$\begin{array}{l} p \uparrow n \downarrow \\ p \downarrow n \uparrow \end{array} \quad \frac{d}{dt} p^k = \frac{\Delta\omega^2}{d^2} \cos(4\omega t + 2\epsilon) \hat{d}^k \quad (30)$$

$$\begin{array}{l} p \uparrow n \uparrow \\ p \downarrow n \downarrow \end{array} \quad \begin{array}{l} \frac{d}{dt} p^k = \frac{\Delta\omega^2}{d^2} [\hat{d}^k + 4r^k \hat{r} \cdot \hat{d} - 6\hat{d}^k (\hat{r} \cdot \hat{d})^2] \\ \hat{r}^k = (-\sin(2\omega t + \epsilon), \cos(2\omega t + \epsilon), 0) \end{array} \quad (31)$$

$$\begin{array}{l} p \downarrow n \downarrow \end{array} \quad \begin{array}{l} \frac{d}{dt} p^k = \frac{\Delta\omega^2}{d^2} [\hat{d}^k + 4r^k \hat{r} \cdot \hat{d} - 6\hat{d}^k (\hat{r} \cdot \hat{d})^2] \\ \hat{r}^k = (-\sin(2\omega t + \epsilon), -\cos(2\omega t + \epsilon), 0) \end{array} \quad (32)$$

where $\epsilon = \phi_0^A + \phi_0^B$. For those scattering states, the right hand side of the equations oscillates very quickly and the scattering angle is suppressed by the very small factor

$$\sim e^{-(\frac{\omega\gamma}{v})} \quad (33)$$

since ω and γ are large while v is small.

We underline that this is the first analytical calculation of nucleon-nucleon scattering from essentially first principles, without recourse to *ad hoc* models or potentials. We emphasize that the Skyrme model is in principle derivable from QCD and f_π and e are, as such, calculable parameters and, in that sense this is also a QCD calculation. To calculate the classical scattering cross-section we need to compute the scattering for all different polarizations relative to the initial scattering plane. This would comprise a different project which would probably be best achieved by numerical methods. Therefore we are unable at this point to make a direct comparison with the experiment.

A peculiar aspect of our treatment is the strong dependence of the scattering angle on the variables δ and ϵ which are just the phase lag between the rotation of the Skyrmions at some time t before the scattering. These act as hidden variables, and have to be measured after the scattering or averaged over.

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